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# Classical Yang–Mills field equations coupled to a scalar triplet invariant under subgroups of the conformal group

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**Abstract**

Classical Yang–Mills-Scalar field equations on Minkowski space with gauge group $SU(2)$ are determined. Solutions invariant up to a gauge transformation under a four-dimensional subgroup of the conformal group are determined for the case of a scalar matter triplet field. © 1999 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Classical Yang–Mills equations have generated considerable interest over the years both from a mathematical point of view, since in their most general form, they represent a system of nonlinear partial differential equations, and in the application of their solutions to physical problems such as those that arise in physics applications. This has been especially apparent in the area of particle physics, where the classical regime serves as a background for the quantized versions of a theory. One can consider solving these field equations for solutions on their own with no external couplings to matter fields, such as scalar or spinor fields, as done in [1]. There has been considerable interest in these equations coupled to a matter triplet and doublet as well as spinor fields [2–5]. A complete solution of this task is likely not possible. However, one can impose symmetry conditions on the problem. For example, one may consider invariance of the solutions under four-dimensional subgroups

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of the conformal group $C(3,1)$ of Minkowski space [6,7]. Some work along these lines for spinor fields has been done for some subgroups of $C(3,1)$ in [8]. The maximal symmetry group of the Yang–Mills system is the conformal group.

Here, we consider the problem of obtaining the Yang–Mills equations on Minkowski space, or its conformal compactification. The field equations for a system of $SU(2)$ gauge fields are formulated such that there is a coupling to a scalar triplet in a pseudo-Riemannian space. Four successive subgroups of $C(3,1)$ will be considered. The analysis of the system has been carried out with respect to the following four subgroups of $C(3,1)$, namely, $SO(2,1)\times SO(1,1)$, $SO(3)\times SO(1,1)$, $SO(2,1)\times SO(2)$ and $SO(3)\times SO(2)$. One can obtain the field equations for the gauge field and, introducing a representation for the scalar field on the right hand side, one obtains the coupled Yang–Mills matter system. There is, in addition, a scalar field equation which determines the dynamics of the scalar field with self-coupling. It will be shown that there is enough simplicity at the subgroup level to allow one to obtain solutions for the scalar field. In fact, it is shown that the only solution for the scalar field is the zero solution for each of these subgroups, unless as in some cases one of the components of the gauge potential is taken to be zero. In these cases there could be a nonzero solution to the system, but with the gauge field decoupled from the matter field.

## 2. Mathematical formulation

Let $\mathcal{M}$ be a pseudo-Riemannian space, $\{l_\mu\}$ a basis of vector fields on $\mathcal{M}$ and $\{\theta^\mu\}$ the dual basis. Suppose $\mathcal{M}$ has the metric

$$g=\tfrac{1}{2}g_{\mu\nu}(\theta^\mu\otimes\theta^\nu+\theta^\nu\otimes\theta^\mu).$$

Let $H$ be the gauge group, $\tilde{h}$ its Lie algebra. The gauge potential is defined by a set of $\tilde{h}$-valued one-forms $\{\omega_\alpha\}$ on some open covering $\{U_\alpha\}$ of $\mathcal{M}$, and the gauge fields are the components of the corresponding curvature two-form. We will write

$$\omega=\omega_\mu\,\mathrm{d}x^\mu,\qquad \omega_\mu=\omega_\mu^\alpha t_\alpha\in\tilde{h},$$

where $\{t_\alpha\}$ is a basis of the Lie algebra $\tilde{h}$. Denote by $\Phi$, the triplet of scalar fields in the adjoint representation of $SU(2)$, and we shall take as a basis for $su(2)$ the set $\{t_a=\sigma_a/2i, a=1,2,3\}$, where $\sigma_a$ are the standard Pauli matrices. In terms of its components, $\Phi$ will read

$$\Phi=\phi^a t_a.$$

The field equations on Minkowski space are given by

$$^*D^*D\omega=[\Phi,D\Phi],\qquad {}^*D^*D\Phi=\lambda|\Phi|^2\Phi,\quad |\Phi|^2=\phi^a\phi_a,\tag{1}$$

where $D$ stands for the exterior covariant derivative

$$D=\mathrm{d}+[\omega,\cdot],\tag{2}$$

where the gauge coupling constant has been scaled to unity. The star operator will be defined with respect to the metric of the relevant subspace.

The field intensity corresponding to $\omega$ is the two-form

$$F = D\omega = \mathrm{d}\omega + \tfrac{1}{2}[\omega, \omega], \tag{3}$$

which effectively appears on the left-hand side of (1). Recall the definition of the invariance up to gauge transformation. Let $G$ be a Lie transformation group, with Lie algebra $\tilde{g}$, acting on the manifold $\mathcal{M}$,

$$f : G \times \mathcal{M} \to \mathcal{M}, \quad f(g, x) = f_g(x).$$

Denoting by $f_g^*\omega_\alpha$ and $f_g^*\Phi_\alpha$, the pull backs of $\omega_\alpha$ and $\Phi_\alpha$ under $f_g$ on the open set $U_\alpha$, we shall say that these potentials and fields are invariant under $G$ if and only if

$$f_g^*\omega_\alpha = A\,\mathrm{d}\rho_\alpha(g, x)^{-1}\omega_\alpha + \rho_\alpha^{-1}(g, x)\,\mathrm{d}\rho_\alpha(g, x),$$

and

$$f_g^*\Phi_\alpha = \mathcal{D}[\rho_\alpha^{-1}(g, x)]\Phi_\alpha,$$

for all $g \in G$, where $\mathcal{D}$ is a representation of the gauge group on a vector space $V$, $\mathcal{D} : H \to GL(V)$. The matter fields transforming according to that representation are defined as sets of $V$-valued fields $\Phi_\alpha$ on $U_\alpha$ related by $\Phi_\beta = \mathcal{D}(k_{\alpha\beta}^{-1})\Phi_\alpha$ on $U_\alpha \cap U_\beta$. Here, $\rho : G \times U_\alpha \to H$ defines a gauge transformation.

In this paper, we use invariant fields with invariance discussed in [3], on open sets $U$ such that $U = V \times G/G_o$, where $V$ is a one-dimensional contractible manifold called the transverse manifold, and the isotropy group $G_o$ is the same for every $t \in V$. The invariant gauge potentials are of the form,

$$\omega = \mu + W \circ \sigma^{-1}\,\mathrm{d}\sigma,$$

where $W : V \to Hom(\tilde{g}, \tilde{h})$ is the Wang map satisfying the following conditions

$$W(\xi) = \lambda_*(\xi),$$

for all $\xi \in \tilde{g}_o$ and

$$W(A\,\mathrm{d}g^{-1}\eta) = A\,\mathrm{d}\lambda(g)^{-1}(W(\eta)),$$

for all $\eta \in \tilde{g}, g \in G_o$. In these relations, $\tilde{g}_o$ denotes the Lie algebra of $G_o$ and $\mu$ is a one-form on $V$ which takes values in the Lie algebra of the centralizer of $C^\lambda$ of $\lambda(G_o)$ in $H$. For the case at hand, the manifold $\mathcal{M}$ is Minkowski space $M$ or its conformal compactification $\bar{M}$. The invariant gauge potentials will not be developed here, one should consult [3]. The potentials from there will simply be put into (3) to obtain the field intensity.

## 3. Invariant fields and Yang–Mills-Scalar equations

Let us consider the four subgroups of $C(3, 1)$ namely, $SO(2, 1) \times SO(1, 1)$, $SO(3) \times SO(1, 1)$, $SO(2, 1) \times SO(2)$ and $SO(3) \times SO(2)$ in turn. Let us begin by describing the action of these subgroups as well as the associated metric and gauge field for the subgroups. The gauge field will be determined by the structure of the subgroup and its corresponding metric, as pointed out in [3]. The first case, which will be the more complicated subgroup, namely, $SO(2,1) \times SO(1,1)$, will be considered in more detail first as an example. The other subgroups of interest will be treated briefly, and only results will be presented.

1. Let us first consider the case of the subgroup $SO(2,1) \times SO(1,1)$. The action of $SO(2,1)$ on Minkowski space is induced from its action on $\mathbb{R}^6$. A system of coordinates $(\lambda, \xi, \psi, t)$ is used which is defined by

$$x^0 = e^{\lambda} \cosh\xi \cosh\psi, \quad x^1 = e^{\lambda} \sinh\psi, \quad x^2 = e^{\lambda} \cosh\psi \sinh\xi, \quad x^3 = e^{\lambda} t.$$

The factor $SO(2,1)$ acts on the subspace (012) and $SO(1,1)$ on the subspace $(4,5)$ of $\mathbb{R}^6$. The action of $SO(1,1)$ on $(\eta^4, \eta^5)^T$ is for example, $\eta'^4 = \eta^4 \cosh\lambda + \eta^5 \sinh\lambda$, and $\eta'^5 = \eta^4 \sinh\lambda + \eta^5 \cosh\lambda$. Then under the action of $g_\lambda \in SO(1,1)$, $\eta^4 + \eta^5$ is transformed into $e^{\lambda}(\eta^4 + \eta^5)$, so the action of $g_\lambda$ on Minkowski space $M$ consists of multiplying every vector $x \in M$ by $e^{-\lambda}$. One can write the action of $SO(2,1) \times SO(1,1)$ as a global action of $\mathbb{R} \times SO(2,1)$ on $M$ with $(g_\lambda, g)x = e^{\lambda} g x$ for all $\lambda \in \mathbb{R}$, $g \in SO(2,1)$.

The invariant gauge potential for this subgroup is given by

$$\omega = y\, \mathrm{d}\psi t_1 + y \cosh\psi\, \mathrm{d}\xi t_2 + (u\, \mathrm{d}\lambda + f\, \mathrm{d}t + \sinh\psi\, \mathrm{d}\xi) t_3. \tag{4}$$

From $\omega$, one can calculate $\mathrm{d}\omega$,

$$\begin{aligned}\mathrm{d}\omega = {} & y'\, \mathrm{d}t \wedge \mathrm{d}\psi t_1 + (y' \cosh\psi\, \mathrm{d}t \wedge \mathrm{d}\xi + y \sinh\psi\, \mathrm{d}\psi \wedge \mathrm{d}\xi) t_2 \\ & + (u'\, \mathrm{d}t \wedge \mathrm{d}\lambda + \cosh\psi\, \mathrm{d}\psi \wedge \mathrm{d}\xi) t_3,\end{aligned}$$

and the bracket is given by

$$\begin{aligned}\tfrac{1}{2}[\omega, \omega] = {} & (uy \cosh\psi\, \mathrm{d}\xi \wedge \mathrm{d}\lambda + fy \cosh\psi\, \mathrm{d}\xi \wedge \mathrm{d}\lambda + fy \cosh\psi\, \mathrm{d}\xi \wedge \mathrm{d}t) t_1 \\ & - (uy\, \mathrm{d}\psi \wedge \mathrm{d}\lambda + yf\, \mathrm{d}\psi \wedge \mathrm{d}t + y \sinh\psi\, \mathrm{d}\psi \wedge \mathrm{d}\xi) t_2 \\ & + y^2 \cosh\psi\, \mathrm{d}\psi \wedge \mathrm{d}\xi t_3\end{aligned}$$

The gauge field $F$ is calculated from Eq. (3),

$$\begin{aligned}F = {} & (y'\, \mathrm{d}t \wedge \mathrm{d}\psi + yf \cosh\psi\, \mathrm{d}\xi \wedge \mathrm{d}t - uy \cosh\psi\, \mathrm{d}\lambda \wedge \mathrm{d}\xi) t_1 \\ & + (y' \cosh\psi\, \mathrm{d}t \wedge \mathrm{d}\xi + uy\, \mathrm{d}\lambda \wedge \mathrm{d}\psi - yf\, \mathrm{d}\psi \wedge \mathrm{d}t) t_2 \\ & - (u'\, \mathrm{d}\lambda \wedge \mathrm{d}t + \cosh\psi\, \mathrm{d}\xi \wedge \mathrm{d}\psi + y^2 \cosh\psi\, \mathrm{d}\xi \wedge \mathrm{d}\psi) t_3.\end{aligned} \tag{5}$$

In terms of these variables, the metric can be written in the form,

$$g_{ij} = \begin{bmatrix} -e^{2\lambda}(t^2 - 1) & 0 & 0 & -te^{2\lambda} \\ 0 & -e^{2\lambda} \cosh^2\psi & 0 & 0 \\ 0 & 0 & -e^{2\lambda} & 0 \\ -te^{2\lambda} & 0 & 0 & -e^{2\lambda} \end{bmatrix}$$

which has the determinant $g = \det g_{ij} = -e^{8\lambda} \cosh^2\psi$.

The dual of $F$ can now be calculated with respect to $g_{ij}$ to give,

$$\begin{aligned}{}^*F = {} & [y'(t^2-1)\cosh\psi\, \mathrm{d}\lambda\wedge \mathrm{d}\xi - y't\cosh\psi\, \mathrm{d}\xi\wedge \mathrm{d}t + yf(t^2-1)\,\mathrm{d}\lambda\wedge \mathrm{d}\psi \\ & - yft\,\mathrm{d}\psi\wedge \mathrm{d}t + uy\,\mathrm{d}\psi\wedge \mathrm{d}t + uyt\,\mathrm{d}\psi\wedge \mathrm{d}\lambda]t_1 + [-y'(t^2-1)\,\mathrm{d}\lambda\wedge \mathrm{d}\psi \\ & + y't\,\mathrm{d}\psi\wedge \mathrm{d}t - uy\cosh\psi\,\mathrm{d}t\wedge \mathrm{d}\xi + uyt\cosh\psi\,\mathrm{d}\xi\wedge \mathrm{d}\lambda \\ & + yf(t^2-1)\cosh\psi\,\mathrm{d}\lambda\wedge \mathrm{d}\xi - yft\cosh\psi\,\mathrm{d}\xi\wedge \mathrm{d}t]t_2 \\ & + [u'\cosh\psi\,\mathrm{d}\xi\wedge \mathrm{d}\psi - (y^2+1)\,\mathrm{d}\lambda\wedge \mathrm{d}t]t_3.\end{aligned}$$

Differentiating this once more with respect to $D$ given in (2), and writing the field equation as

$$D^*F = {}^*J, \tag{6}$$

one will produce a system of equations which are generated from the coefficients of the $t_i$. From the coefficient of $t_1$, the coefficients of the forms $\mathrm{d}t\wedge \mathrm{d}\lambda\wedge \mathrm{d}\xi$ and $\mathrm{d}t\wedge \mathrm{d}\lambda\wedge \mathrm{d}\psi$ give the pair of equations,

$$(y'(t^2-1))' + (f^2+y^2+1)y - (ft-u)^2 y$$

and

$$y^{-1}((y^2 f(t^2-1)) - uty^2)'.$$

From $t_2$, one has

$$-((y'(t^2-1))' + 2uyft - yf^2(t^2-1) - u^2 y + y(y^2+1))$$

and

$$y^{-1}(y^2 f(t^2-1) - uty^2)'\cosh\psi$$

and from $t_3$, one obtains the pair

$$(u'' + 2y^2(ft-u))\cosh\psi$$

and

$$-2y^2(f(t^2-1) - tu)\cosh\psi.$$

All the other coefficients vanish. These equations have to be matched with the corresponding $t_i$ and three-forms which appear in the matter current on the right-hand side of the field equations. For example, if one takes the current $J$ to be identically zero, so there is no coupling to the matter current, one would set the six equations above to zero, and try to obtain solutions of the resulting system. By calculating the matter current, we can then match the corresponding coefficients of the forms to obtain the full set of equations coupled to the scalar field.

If one takes the form for $\Phi$ given above, writing $\phi_a t_a$ for notational convenience and takes the $\phi_a$ to be functions of only the $t$-variable in the same way that the $u$, $y$ and $f$ functions are, the required bracket can be written in the following form

$$
\begin{aligned}
[\Phi, D\Phi] = {} & ((\phi_2\phi_3' - \phi_3\phi_2') - f\phi_1\phi_3)\,\mathrm{d}t + (\phi_2^2 + \phi_3^2)y\,\mathrm{d}\psi \\
& -(\phi_1\phi_3 \sinh\psi + y\phi_1\phi_2\cosh\psi)\,\mathrm{d}\xi - u\phi_1\phi_3\,\mathrm{d}\lambda)t_1 \\
& +((\phi_3\phi_1' - \phi_1\phi_3' - f\phi_2\phi_3)\,\mathrm{d}t + ((\phi_1^2 + \phi_3^2)y\cosh\psi - \phi_2\phi_3\sinh\psi)\,\mathrm{d}\xi \\
& -u\phi_2\phi_3\,\mathrm{d}\lambda - y\phi_1\phi_2\,\mathrm{d}\psi)t_2 + (\phi_1\phi_2' - \phi_2\phi_1' + f(\phi_1^2 + \phi_2^2))\,\mathrm{d}t \\
& +(\phi_1^2 + \phi_2^2)u\,\mathrm{d}\lambda + ((\phi_1^2 + \phi_2^2)\sinh\xi - y\phi_2\phi_3\cosh\psi)\,\mathrm{d}\xi - y\phi_1\phi_3\,\mathrm{d}\psi)t_3,
\end{aligned}
$$

where the components of $\Phi$ have been written in the form $\phi_i$ for notational convenience. Taking the dual of this with respect to the metric and collecting the coefficients of the individual three forms yield the following system of twelve equations, four from the coefficients of each $t_i$, as follows,

$$(y'(t^2 - 1))' + (f^2 + y^2 + 1)y - (ft - u)^2 y = -y(\phi_2^2 + \phi_3^2)e^{2\lambda}\cosh\psi, \tag{7}$$

$$y^{-1}((y^2 f(t^2 - 1) - uty^2)' = -e^{2\lambda}\phi_1(\phi_3\tanh\psi + y\phi_2), \tag{8}$$

$$((\phi_2\phi_3' - \phi_3\phi_2' - f\phi_1\phi_3)(t^2 - 1) + ut\phi_1\phi_3)e^{2\lambda}\cosh\psi = 0 \tag{9}$$

$$((\phi_2\phi_3' - \phi_3\phi_2' - f\phi_1\phi_3)t + u\phi_1\phi_3)e^{2\lambda}\cosh\psi = 0, \tag{10}$$

$$
\begin{aligned}
& -((y'(t^2 - 1))' + 2uyft - yf^2(t^2 - 1) - u^2 y + y(y^2 + 1)) \\
& \quad = e^{2\lambda}((\phi_1^2 + \phi_3^2)y - \phi_2\phi_3\tanh\psi),
\end{aligned} \tag{11}
$$

$$(y^2 f(t^2 - 1) - uty^2)' = -y^2\phi_1\phi_2 e^{2\lambda}, \tag{12}$$

$$e^{2\lambda}\cosh\psi((\phi_3\phi_1' - \phi_1\phi_3' + f\phi_2\phi_3)(t^2 - 1) + ut\phi_2\phi_3) = 0, \tag{13}$$

$$e^{2\lambda}\cosh\psi((\phi_3\phi_1' - \phi_1\phi_3' + f\phi_2\phi_3)t + u\phi_2\phi_3) = 0, \tag{14}$$

$$
\begin{aligned}
& (u'' + 2y^2(ft - u))\cosh\psi = -(t(\phi_1\phi_2' - \phi_2\phi_1' + f(\phi_1^2 + \phi_2^2)) \\
& \quad -u(\phi_1^2 + \phi_2^2))e^{2\lambda}\cosh\psi,
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
& -2y^2(f(t^2 - 1) - ut)\cosh\psi = ((\phi_1\phi_2' - \phi_2\phi_1' + f(\phi_1^2 + \phi_2^2))(t^2 - 1) \\
& \quad -ut(\phi_1^2 + \phi_2^2))e^{2\lambda}\cosh\psi,
\end{aligned} \tag{16}
$$

$$(\phi_1^2 + \phi_2^2)\sinh\psi - y\phi_2\phi_3\cosh\psi = 0, \tag{17}$$

$$e^{2\lambda}\cosh\psi\, y\phi_1\phi_3 = 0. \tag{18}$$

The dynamics of the scalar field is determined by the equation

$$D^* D\Phi = \lambda|\Phi|^2\Phi\,{}^*1, \tag{19}$$

and one obtains from this three additional equations for the matter fields,

$$
\begin{aligned}
& e^{2\lambda}\cosh\psi([(\phi_1' - f\phi_2)(t^2 - 1) - ut\phi_2]' + u[-(\phi_2' + f\phi_1)t + u\phi_1] \\
& \quad + f[(\phi_2' + f\phi_1)(t^2 - 1) + ut\phi_1] - \phi_1\tanh^2\psi + y^2\phi_2) = \lambda\sigma_1|\Phi|^2\phi_1,
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
& e^{2\lambda}\cosh\psi([(\phi_2' + f\phi_1)(t^2 - 1) + ut\phi_1]' - u[(\phi_1' - f\phi_2)t + u\phi_2] \\
& \quad + f[(\phi_1' + f\phi_2)(t^2 - 1) - ut\phi_2] + (\phi_2 - y\phi_3\tanh\psi) + y^2\phi_2) = \lambda\sigma_1|\Phi|^2\phi_2,
\end{aligned} \tag{21}
$$

$$e^{2\lambda}\cosh\psi((\phi_3'(t^2 - 1))' + y\phi_2\tanh\psi) = \lambda\sigma_1|\Phi|^2\phi_3, \tag{22}$$

The factors $\sigma_i$, where $i = 1, \ldots, 4$ that appear on the right of the matter equations depends on the coordinates and comes from the $^*1$ operation for each of the subgroups considered here.

2. The action of $SO(2,1) \times SO(2)$ in $\mathbb{R}^6$ has the factor $SO(2,1)$ acting on the subspace (345) and $SO(2)$ on the subspace (12). It is simpler to work in the conformally compactified space, the solutions on Minkowski space are then obtained by pull-back.

For this subgroup, the invariant gauge potential is given by

$$\omega = y\,\mathrm{d}\psi t_1 + y\cosh\psi\,\mathrm{d}\xi t_2 + (u\,\mathrm{d}\lambda + f\,\mathrm{d}t + \sinh\psi\,\mathrm{d}\xi)t_3. \tag{23}$$

Using Eq. (3), one obtains the following expression for $F$,

$$\begin{aligned} F = {} & [y'\,\mathrm{d}t \wedge \mathrm{d}\psi - uy\cosh\psi\,\mathrm{d}\lambda \wedge \mathrm{d}\xi - yf\cosh\psi\,\mathrm{d}t \wedge \mathrm{d}\xi]t_1 \\ & + [y'\cosh\psi\,\mathrm{d}t \wedge \mathrm{d}\xi + yf\,\mathrm{d}t \wedge \mathrm{d}\psi + uy\,\mathrm{d}\lambda \wedge \mathrm{d}\psi]t_2 \\ & + [u'\,\mathrm{d}t \wedge \mathrm{d}\lambda + (y^2+1)\cosh\psi\,\mathrm{d}\psi \wedge \mathrm{d}\xi]t_3. \end{aligned} \tag{24}$$

The matter current can be written in the following form,

$$\begin{aligned} [\Phi, D\Phi] = {} & ((\phi_2\phi_3' - \phi_3\phi_2' - f\phi_1\phi_3)\,\mathrm{d}t - \phi_1(\phi_3\sinh\psi + y\phi_2\cosh\psi)\,\mathrm{d}\xi \\ & + y(\phi_2^2 + \phi_3^2)\,\mathrm{d}\psi - u\phi_1\phi_3\,\mathrm{d}\lambda)t_1 + ((\phi_3\phi_1' - \phi_1\phi_3' - f\phi_2\phi_3)\,\mathrm{d}t \\ & - u\phi_2\phi_3\,\mathrm{d}\lambda - y\phi_1\phi_2\,\mathrm{d}\psi + ((\phi_1^2 + \phi_3^2)y\cosh\psi - \psi_2\psi_3\sinh\psi)\,\mathrm{d}\xi)t_2 \\ & + ((\phi_1\phi_2' - \phi_2\phi_1' + f(\phi_1^2 + \phi_2^2))\,\mathrm{d}t + u(\phi_1^2 + \phi_2^2)\,\mathrm{d}\lambda \\ & - y\phi_1\phi_3\,\mathrm{d}\psi + (\phi_1^2 + \phi_2^2)\sinh\psi\cosh\psi\,\mathrm{d}\xi)t_3. \end{aligned}$$

To substitute this into the first set of field equations, one can use the form (6), and take duals with respect to the metric $g = (1, -1, -\cos^2 t, -\cos^2 t\cosh^2\psi)$. This leads to the following system of equations, which are summarized below,

$$(\phi_2\phi_3' - \phi_3\phi_2' - f\phi_1\phi_3)\cos^2 t\cosh\psi = 0 \tag{25}$$

$$(f'y + 2y'f) = -(\phi_3\tanh\psi + y\phi_2)\phi_1, \tag{26}$$

$$u\phi_1\phi_3\cos^2 t\cosh\psi = 0 \tag{27}$$

$$(y'' + u^2 y - yf^2 + y(y^2+1)\cos^{-2} t)\cosh\psi = -y(\phi_2^2 + \phi_3^2)\cosh\psi, \tag{28}$$

$$(\phi_3\phi_1' - \phi_1\phi_3' - f\phi_2\phi_3)\cos^2 t\cosh\psi = 0, \tag{29}$$

$$u\phi_2\phi_3\cos^2 t\cosh\psi = 0, \tag{30}$$

$$(f'y + 2y'f) = y\phi_1\phi_2, \tag{31}$$

$$-(y'' + u^2 y - f^2 y + y(y^2+1)\cos^{-2} t) = ((\phi_1^2 + \phi_3^2)y - \phi_2\phi_3\tanh\psi), \tag{32}$$

$$((u'\cos^2 t)' + 2uy^2)\cosh\psi = -u(\phi_1^2 + \phi_2^2)\cos^2 t\cosh\psi, \tag{33}$$

$$2fy^2\cosh\psi = -(\phi_1\phi_2' - \phi_2\phi_1' + f(\phi_1^2 + \phi_2^2)), \tag{34}$$

$$y\phi_1\phi_3\cosh\psi = 0, \tag{35}$$

$$(\phi_1^2 + \phi_2^2)\sinh\psi - y\phi_2\phi_3\cosh\psi = 0, \tag{36}$$

$$\begin{aligned} & ((\phi_1' - f\phi_2)\cos^2 t)' - f(\phi_2' + f\phi_1)\cos^2 t + u^2\phi_1\cos^2 t + \phi_1 \\ & \quad - y^2\phi_2 = \lambda\sigma_2|\Phi|^2\phi_1, \end{aligned} \tag{37}$$

$$((\phi_2' - f\phi_1)\cos^2 t)'\cosh\psi + f(\phi_1' - f\phi_2)\cos^2 t\cosh\psi + u^2\phi_2\cos^2 t\cosh\psi$$
$$-y\sinh\psi\phi_3 + (\phi_2 + y^2\phi_2)\cosh\psi = \lambda\sigma_2|\Phi|^2\phi_2, \tag{38}$$
$$((\phi_3'\cos^2 t)')\cosh\psi - y\phi_2\sinh\psi + 2y^2\phi_3\cosh\psi = \lambda\sigma_2|\Phi|^2\phi_3. \tag{39}$$

3. For the subgroup $SO(3)\times SO(1,1)$, the subgroup $SO(3)$ acts on the subspace (123) and $SO(1,1)$ on the subspace (4, 5) of $\mathbb{R}^6$. Then $SO(3)$ may be considered as the subgroup of the Lorentz group acting on the subspace (1, 2, 3). The only generic stratum is the orbit of the subset $V$ defined by $x^1 = 1, x^2 = x^3 = 0$, which will be taken as a transverse section. With respect to the coordinate system for this subgroup, $x^0 = e^\lambda t$, $x^1 = e^\lambda\cos\xi\cos\psi$, $x^2 = e^\lambda\sin\psi$, $x^3 = e^\lambda\sin\xi\cos\psi$, the invariant gauge potential is given by

$$\omega = (u\,\mathrm{d}\lambda + f\,\mathrm{d}t + \sin\psi\,\mathrm{d}\xi)t_1 - y\,\mathrm{d}\psi t_2 + y\cos\psi\,\mathrm{d}\xi t_3, \tag{40}$$

and the gauge field is,

$$F = (u'\,\mathrm{d}t\wedge\mathrm{d}\lambda - (y^2-1)\cos\psi\,\mathrm{d}\psi\wedge\mathrm{d}\xi)t_1$$
$$-(y'\,\mathrm{d}t\wedge\mathrm{d}\psi + yf\cos\psi\,\mathrm{d}t\wedge\mathrm{d}\xi + y\cos\psi\,\mathrm{d}\lambda\wedge\mathrm{d}\xi)t_2$$
$$+(y'\cos\psi\,\mathrm{d}t\wedge\mathrm{d}\xi - yf\,\mathrm{d}t\wedge\mathrm{d}\psi - uy\,\mathrm{d}\lambda\wedge\mathrm{d}\psi)t_3. \tag{41}$$

From Eq. (6) for the gauge field and Eq. (19) for the matter field, by matching coefficients of $t_i$ on both sides of these and then the coefficients of the forms on both sides, one obtains the following system of equations,

$$-(u'' + 2uy^2 - 2tfy^2)\cos\psi$$
$$= -(\phi_2\phi_3' - \phi_3\phi_2' + (f-u)(\phi_2^2+\phi_3^2))e^{2\lambda}\cos\psi, \tag{42}$$
$$2y^2(f(t^2-1) - ut)\cos\psi$$
$$= -(\phi_2\phi_3' - \phi_3\phi_2' + (f-ut)(\phi_2^2+\phi_3^2))e^{2\lambda}\cos\psi, \tag{43}$$
$$y\phi_1\phi_2 e^{2\lambda}\cos\psi = 0, \tag{44}$$
$$(\phi_2^2+\phi_3^2)e^{2\lambda}\tan\psi = 0, \tag{45}$$
$$((\phi_3\phi_1' - \phi_1\phi_3' - f\phi_1\phi_2)t + u\phi_1\phi_2)e^{2\lambda}\cos\psi = 0, \tag{46}$$
$$((\phi_3\phi_1' - \phi_1\phi_3' - f\phi_1\phi_2)(t^2-1) + ut\phi_1\phi_2)e^{2\lambda}\cos\psi = 0, \tag{47}$$
$$((y'(t^2-1))' + 2uytf - u^2y - yf^2(t^2-1) - y(y^2-1))\cos\psi$$
$$= y(\phi_1^2+\phi_3^2)e^{2\lambda}\cos\psi, \tag{48}$$
$$y^{-1}(y^2f(t^2-1) - uy^2t)' = e^{2\lambda}\phi_2(\phi_1\tan\psi + y\phi_3), \tag{49}$$
$$((y'(t^2-1))' + (tf-u)uy - f(f(t^2-1)-ut)y - (y^2-1)y)$$
$$= e^{2\lambda}(y(\phi_1^2+\phi_2^2) - \phi_1\phi_3\tan\psi) \tag{50}$$
$$-(fy^2(t^2-1) - uy^2t)'y^{-1}\cos\psi = y\phi_2\phi_3e^{2\lambda}\cos\psi, \tag{51}$$
$$e^{2\lambda}\cos\psi((\phi_1\phi_2' - \phi_2\phi_1' - f\phi_1\phi_3)(t^2-1) + ut\phi_1\phi_3) = 0, \tag{52}$$
$$(-(\phi_1'(t^2-1))' + y(2y\phi_1 - \phi_3\tan\psi))e^{2\lambda}\cos\psi = \lambda\sigma_3|\Phi|^2\phi_1, \tag{53}$$

$$(-((\phi_2' - f\phi_3)(t^2-1))'\cos\psi - (ut\phi_3)'\cos\psi + u(-(\phi_3' + f\phi_2)t + u\phi_2)\cos\psi$$
$$+f((\phi_3' + f\phi_2)(t^2-1) - ut\phi_2)\cos\psi + \sin\psi\tan\psi\,\phi_2 + y^2\phi_2\cos\psi)e^{2\lambda}$$
$$= \lambda\sigma_3|\Phi|^2\phi_2, \tag{54}$$
$$[-((\phi_3' + f\phi_2)(t^2-1))' + (ut\phi_2)' + u((\phi_2' - f\phi_3)t + u\phi_3)$$
$$+f((\phi_2' - f\phi_3)(t^2-1) + ut\phi_3) - (y\phi_1 - \phi_3\tan\psi)\tan\psi + y^2\phi_3]e^{2\lambda}\cos\psi$$
$$= \lambda\sigma_3|\Phi|^2\phi_3. \tag{55}$$

4. Finally, the action of $SO(3)\times SO(2)$ has the factor $SO(3)$ acting on the subspace (123) and $SO(2)$ acting on the subspace (05) of $\mathbb{R}^6$. Again, the calculations are simpler in conformally compactified space and the solutions on Minkowski space are obtained by pull-back.

The invariant gauge potentials and fields are given respectively by

$$\omega = (u\,\mathrm{d}\lambda + f\,\mathrm{d}t + \sin\psi\,\mathrm{d}\xi)t_1 - y\,\mathrm{d}\psi t_2 + y\cos\psi\,\mathrm{d}\xi t_3, \tag{56}$$

and $F$ is given by the expression,

$$F = (u'\,\mathrm{d}t\wedge\mathrm{d}\lambda - (y^2-1)\cos\psi\,\mathrm{d}\psi\wedge\mathrm{d}\xi)t_1$$
$$+(-y'\,\mathrm{d}t\wedge\mathrm{d}\psi + uy\cos\psi\,\mathrm{d}\xi\wedge\mathrm{d}\lambda + yf\cos\psi\,\mathrm{d}\xi\wedge\mathrm{d}t)t_2$$
$$+(y'\cos\psi\,\mathrm{d}t\wedge\mathrm{d}\xi + uy\,\mathrm{d}\psi\wedge\mathrm{d}\lambda + yf\,\mathrm{d}\psi\wedge\mathrm{d}t)t_3. \tag{57}$$

Here, we will just summarize the resulting field equations. Using (6) and (18), one obtains the system,

$$((u'\cos^2 t)' - 2uy^2)\cos\psi = u(\phi_2^2 + \phi_3^2)\cos^2 t\cos\psi, \tag{58}$$
$$2y^2 f\cos\psi = -(\phi_2\phi_3' - \phi_3\phi_2' + f(\phi_2^2+\phi_3^2))\cos^2 t\cos\psi, \tag{59}$$
$$y\phi_1\phi_2\cos\psi = 0, \tag{60}$$
$$(\phi_2^2+\phi_3^2)\tan\psi = 0, \tag{61}$$
$$(y'' - \frac{y}{\cos^2 t}(y^2-1) + (u^2 - f^2)y)\cos\psi = -y(\phi_1^2+\phi_3^2)\cos\psi, \tag{62}$$
$$y^{-1}(y^2 f)' = -(\phi_1\tan\psi + y\phi_3)\phi_2, \tag{63}$$
$$u\phi_1\phi_2\cos^2 t\cos\psi = 0, \tag{64}$$
$$(\phi_3\phi_1' - \phi_1\phi_3' - f\phi_1\phi_2)\cos^2 t\cos\psi = 0, \tag{65}$$
$$-(y'' - \frac{y}{\cos^2 t}(y^2-1) + (u^2-f^2)y) = (y(\phi_1^2+\phi_2^2) - \phi_1\phi_3\tan\psi), \tag{66}$$
$$-y^{-1}(fy^2)'\cos\psi = y\phi_2\phi_3\cos\psi, \tag{67}$$
$$u\phi_1\phi_3\cos^2 t\cos\psi = 0, \tag{68}$$
$$(\phi_1\phi_2' - \phi_2\phi_1' - f\phi_1\phi_3)\cos^2 t\cos\psi = 0, \tag{69}$$
$$-(\phi_1'\cos^2 t)'\cos\psi - y\phi_3\sin\psi = \lambda\sigma_4|\Phi|^2\phi_1, \tag{70}$$
$$((\phi_2' - f\phi_3)\cos^2 t)'\cos\psi - u^2\phi_2\cos^2 t\cos\psi + f(\phi_3' - f\phi_2)\cos^2 t\cos\psi$$
$$-\phi_2\sin\psi\tan\psi - y^2\phi_2\cos\psi = \lambda\sigma_4|\Phi|^2\phi_2, \tag{71}$$

$$((\phi_3' + f\phi_2)\cos^2 t)' \cos\psi - u^2\phi_3 \cos^2 t \cos\psi - f(\phi_2' - f\phi_3)\cos^2 \cos\psi$$
$$+y\phi_1 \sin\psi + \phi_3 \sin\psi \tan\psi + y^2\phi_3 \cos\psi = \lambda\sigma_4|\Phi|^2\phi_3. \quad (72)$$

This completes the derivation of the systems of equations for the subgroups.

## 4. Analysis of equations for subgroups

1. Consider first the subgroup $SO(2,1) \times SO(1,1)$, where the system of field equations are given by (7)–(22). Specific values of the coordinates which might satisfy the system will not be treated as a solution here. One of these equations is quite restrictive as to the type of admissible solutions for the field variables, namely (18),

$$y\phi_1\phi_3 = 0.$$

Let us first assume that the function $y$ is not identically the zero solution. To satisfy (18), let us first take $\phi_3 = 0$. Then setting $\phi_3 = 0$ in Eq. (22) gives the constraint $y\phi_2 = 0$, so if $y \neq 0$, then we must have $\phi_2 = 0$. Putting $\phi_2 = \phi_3 = 0$ in Eq. (17) implies that $\phi_1 = 0$, and we are left with the solution $\Phi_0 \equiv (0,0,0)$ for the scalar field, and $u$, $y$ and $f$ are then determined from the remaining equations with zero on the right-hand side.

Another way to satisfy Eq. (18) is to set $\phi_1 = 0$. If we put $\phi_1 = 0$ in Eq. (14), we obtain

$$\phi_2\phi_3(ft + u) = 0,$$

if $u + ft \neq 0$. Since the case in which $\phi_3 = 0$ has already been treated, this leaves us with examining the case $\phi_2 = 0$. Putting $\phi_1 = \phi_2 = 0$ forces $y\phi_3^2 = 0$ from (7) and this implies that $\phi_3 = 0$. Again, we are left with only the zero solution, $\Phi_0$.

If one wants to consider the possibility of taking $y = 0$, to satisfy (18), then Eq. (11), forces $\phi_2\phi_3 = 0$, which implies that $\phi_2 = 0$ or $\phi_3 = 0$. The case in which $y = 0$ $\phi_3 = 0$ can be treated, as follows. Setting $y = 0$ in (17) implies that $\phi_1^2 + \phi_2^2 = 0$, which in (16) gives $\phi_2'/\phi_2 = \phi_1'/\phi_1$, which implies that $\phi_2 = C\phi_1$, where $C$ is a constant. Putting this in (17) gives $\phi_1 = \phi_2 = 0$. Now suppose that $y = 0$ and $\phi_2 = 0$. Then Eq. (9) implies that $(f(t^2 - 1) - ut)\phi_1\phi_3 = 0$, and in Eq. (17) $\phi_1^2 = 0$, so $\phi_1 = 0$. Putting $\phi_1 = \phi_2 = 0$ in Eq. (21) implies that the following decoupled coexisting pair remain and can be integrated,

$$u'' = 0, \qquad (\phi_3'(t^2 - 1))' = \lambda\sigma_1|\phi_3|^2\phi_3.$$

2. Consider the case of the subgroup $SO(2,1) \times SO(2)$. Three of the equations in the set are quite restrictive with regard to the functions which appear in them. These Eqs. (27), (30) and (35) are summarized below,

$$u\phi_1\phi_3 \cos^2 t \cosh\psi = 0, \quad u\phi_2\phi_3 \cos^2 t \cosh\psi = 0, \quad y\phi_1\phi_3 \cosh\psi = 0.$$

There are a number of combinations which will satisfy these equations. Consider setting $\phi_3 = 0$, then Eq. (25) is automatically satisfied, Eq. (26) and Eq. (31) in the group reduce to

$$(f'y + 2y'f) = -y\phi_1\phi_2, \quad (f'y + 2y'f) = y\phi_1\phi_2$$

which imply that either $\phi_1$ or $\phi_2$ is zero. Since Eq. (36) reduces to $\phi_1^2 + \phi_2^2 = 0$, setting one of these fields equal to zero fixes the other to be equal to zero.

Without putting $u$ or $y$ equal to zero, one must have $\phi_1 = \phi_2 = 0$ if $\phi_3 \neq 0$. The Eq. (28) reduces to

$$y'' + u^2 y - yf^2 + y(y^2+1)\cos^{-2} t = -y\phi_3^2,$$

so (31) and (33) become,

$$f'y + 2y'f = 0,$$
$$(u'\cos^2 t)' + 2uy = 0.$$

Eq. (38) gives

$$y\phi_3 = 0,$$

and (39) gives,

$$((\phi_3'\cos^2 t)')\cosh\psi + 2y^2\phi_3\cosh\psi = \lambda\sigma_2|\phi_3|^2\phi_3.$$

The equation above implies that $\phi_3 = 0$ or $y = 0$. If $\phi_3 = 0$, we are left with the zero solution $\Phi_0$. However, there is the interesting possibility of taking $y = 0$, and in this case one can satisfy all equations provided $u$ and $\phi_3$ satisfy the following pair,

$$(u'\cos^2 t)' = 0, \quad ((\phi_3'\cos^2 t)')\cosh\psi = \lambda\sigma_2|\phi_3|^2\phi_3.$$

for which one can write solutions in terms of integrals with respect to $t$.

3. For the subgroup $SO(3) \times SO(1,1)$, the most restrictive equation is Eq. (44),

$$y\phi_1\phi_2 = 0.$$

Again, let's suppose first that $y \neq 0$. Suppose that $\phi_1 = 0$ which in (53) gives $y\phi_3 = 0$, so $\phi_3 = 0$. Putting $\phi_3 = 0$ in Eq. (45) implies $\phi_2 = 0$, which means that we have simply the zero solution $\Phi_0$. Putting $\phi_2 = 0$ in the above equation also gives zero. In (45), this implies that $\phi_3 = 0$. Putting $\phi_2 = \phi_3 = 0$ in Eq. (55) gives $y\phi_1 = 0$, which implies that $\phi_1 = 0$ if $y \neq 0$, hence again only $\Phi_0$.

Let's consider the possibility of taking $y = 0$ in the above equation. This forces $\phi_1\phi_3 = 0$, from (50), so either $\phi_1 = 0$ or $\phi_3 = 0$. For the case in which $y = 0$ and $\phi_1 = 0$, (45) implies $\phi_2^2 + \phi_3^2 = 0$ and in (43) gives $\phi_2'/\phi_2 = \phi_3'/\phi_3$, so $\phi_3 = C\phi_2$ and (45) implies that $\phi_2 = \phi_3 = 0$. Consider the case in which $y = 0$, $\phi_3 = 0$. In Eq. (45) implies that $\phi_2 = 0$. This combination satisfies all the equations identically except the following two, from (42),

$$u'' = 0,$$

and the first matter equation, (53),

$$-(\phi_1'(t^2-1))'e^{2\lambda}\cos\psi = \lambda\sigma_3|\Phi|^2\phi_1,$$

but in which there is no coupling with the $u$-field explicitly. These can be integrated with respect to $t$.

4. The subgroup $SO(3) \times SO(2)$ is treated last. The most restrictive equations will be those given by (60), (64) and (68)

$$y\phi_1\phi_2 = 0, \qquad u\phi_1\phi_2 = 0, \qquad u\phi_1\phi_3 = 0.$$

Suppose $y \neq 0$, then to satisfy this system, let us first put $\phi_2 = \phi_3 = 0$. Putting this combination into Eq. (72) gives $y\phi_1 = 0$, so $\phi_1 = 0$ also and we are left with $\Phi_0$. Another way of satisfying these three equations simultaneously is to just put $\phi_1 = 0$, which in (70) implies that $\phi_3 = 0$ if $y \neq 0$. Putting $\phi_1 = \phi_3 = 0$ in Eq. (61) gives $\phi_2^2 = 0$. In fact, taking $\phi_3 = 0$ in above equation and (61) forces $\phi_2 = 0$, which both give in Eq. (72), $\phi_1 = 0$ when $y \neq 0$.

Putting $y = 0$ and $u \neq 0$ in (9) gives $\phi_1\phi_3 = 0$. For the case $y = 0$, $\phi_1 = 0$, (62) implies that $\phi_2^2 + \phi_3^2 = 0$, (59) implies that $\phi_2'/\phi_2 = \phi_3'/\phi_3$, and so, as in the previous cases, one has only the $\Phi_0$ solution. When $\phi_3 = 0$ in (59) or (61), one has $\phi_2 = 0$. Thus, with $y = 0$, the system can be completely satisfied except for the pair of equations, such that $u$ and $\phi_1$ are decoupled,

$$(u' \cos^2 t)' = 0,$$
$$-(\phi_1 \cos^2 t)' \cos\psi = \lambda\sigma_4|\Phi|^2\phi_1.$$

which can be easily integrated.

## References

[1] J.P. Antoine, I. Mahara, Classical Yang–Mills–Dirac Equations: Qualitative analysis of some solutions with noncompact symmetry group, Lett. in Math. Phys. 38 (1996) 237–255.
[2] J. Harnad, S. Schneider, L. Vinet, The Yang–Mills system in compactified Minkowski space; invariance conditions and $SU(2)$ invariant solutions, J. Math. Phys. 20 (1979) 931–942.
[3] J.P. Antoine, L. Dabrowski, I. Mahara, Classical Yang–Mills–Dirac system with conformal symmetry: A geometrical analysis, Mod. Phys. Lett. A 9 (1994) 3431–3444.
[4] P. Bracken, Invariant solutions of the Yang–Mills field equations coupled to a scalar doublet, Lett. in Math. Phys. 47 (1999) 19–31.
[5] L. Vinet, Matter-coupled Yang–Mills system in Minkowski space. I. invariant solutions in the presence of scalar fields, Phys. Rev. D 24 (1981) 3179–3193.
[6] J.P. Antoine, M. Jacques, Classical Yang–Mills fields on Minkowski space with a non-compact invariance group, Class. Quantum Grav. 1 (1984) 431–446.
[7] J.P. Antoine, M. Jacques, Classical Yang–Mills fields with conformal invariance: From exact solutions to qualitative analysis, Class. Quantum Grav. 4 (1987) 181–194.
[8] J. Doneux, Y. Saint-Aubin, L. Vinet, Matter-coupled Yang–Mills system in Minkowski space. II. invariant solutions in the presence of dirac spinor fields, Phys. Rev. D 25 (1982) 484–501